The tail of the maximum of Brownian motion minus a parabola

Piet Groeneboom and Nico M. Temme

*Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands,
p.groeneboom@tudelft.nl; http://dutiosc.twi.tudelft.nl/~pietg/
and CWI, Science Park 123, 1098 XG Amsterdam, The Netherlands,
Nico.Temme@cwi.nl; http://homepages.cwi.nl/~nicot/*

June 24, 2011

Abstract

We analyze the tail behavior of the maximum N of $\{W(t)-t^2 : t \ge 0\}$, where W is standard Brownian motion on $[0,\infty)$ and give an asymptotic expansion for $\mathbb{P}\{N \ge x\}$, as $x \to \infty$. This extends a first order result on the tail behavior, which can be deduced from Hüsler and Piterbarg (1999). We also point out the relation between certain results in Groeneboom (2010) and Janson, Louchard and Martin-Löf (2010).

1 Introduction

The distribution function of the maximum of Brownian motion minus a parabola was studied in the two recent papers Janson, Louchard and Martin-Löf (2010) and Groeneboom (2010), both for one-sided and two-sided Brownian motion. The characterization of the distribution function is somewhat different in the two papers, but both characterizations (unavoidably) involve Airy functions. In this note we address the tail behavior of the distribution, a topic that was not addressed in these papers.

The tail behavior of the maximum plays an important role in certain recent studies on the asymptotic distribution of tests for monotone hazards, based on integral-type statistics measuring the distance between the empirical cumulative hazard function and its greatest convex minorant, for example in Groeneboom and Jongbloed (2010).

Let N be defined by

$$N = \max_{t\ge 0}\{W(t)-t^2\}, \tag{1.1}$$

where W is standard Brownian motion on $[0,\infty)$. It can be deduced from Theorem 2.1 in Hüsler and Piterbarg (1999) that the distribution function F_N of N satisfies:

$$1-F_N(x) \sim \frac{1}{\sqrt{3}}\exp\left\{-\frac{8x^{3/2}}{3\sqrt{3}}\right\},\ x\to\infty.$$

In section 2 we will give an asymptotic expansion of the left-hand side, which extends this result. The proof is based on an integral expression for the density, derived from Groeneboom (2010)

MSC 2010 *subject classifications.* 60J65,60J75
Key words and phrases. Brownian motion, parabolic drift, maximum, Airy functions

(which in turn relies on Groeneboom (1989)), and uses a saddle point method for the integral over a shifted path in the complex plane. As a side effect, it also leads to a clarification of the relation between the representations of the distribution, given in Janson, Louchard and Martin-Löf (2010) and Groeneboom (2010).

2 Main results

In the following, we will use Corollary 2.1 of Groeneboom (2010), which is stated below for ease of reference, specialized to the density of the maximum of $W(t)-t^2$ (instead of the more general $W(t)-ct^2$).

Lemma 2.1 (Corollary 2.1 in Groeneboom (2010)) *The density f of N is given by:*

$$f_N(x)=2^{2/3}\left\{\mathrm{Ai}\big(2^{2/3}x\big)-2\,\mathrm{Re}\left(e^{-i\pi/6}\int_0^\infty\frac{\mathrm{Ai}\left(e^{-i\pi/6}u\right)\mathrm{Ai}'\big(iu+2^{2/3}x\big)}{\mathrm{Ai}(iu)}\,du\right)\right\},\ x>0. \tag{2.2}$$

where Ai *is the Airy function* Ai*, as defined in, e.g.,* Olver (2010)[1].

We deduce from this the following representation which is better suited for our purposes.

Lemma 2.2 *The density f_N of N is given by:*

$$f_N(x)=\frac{2^{2/3}}{\pi}\,\mathrm{Re}\left(\int_0^\infty\frac{\mathrm{Ai}\big(iu+2^{2/3}x\big)}{\mathrm{Ai}(iu)^2}\,du\right)=\frac{1}{2^{1/3}\pi}\int_{-\infty}^\infty\frac{\mathrm{Ai}\big(iu+2^{2/3}x\big)}{\mathrm{Ai}(iu)^2}\,du,\ x>0. \tag{2.3}$$

Proof. Integration by parts of the second term of (2.2) yields:

$$f_N(x)=2\cdot2^{2/3}\,\mathrm{Re}\left(\int_0^\infty\mathrm{Ai}\big(iu+2^{2/3}x\big)\frac{d}{du}\left\{\frac{e^{-i\pi/6}\mathrm{Ai}\left(e^{-i\pi/6}u\right)}{i\mathrm{Ai}(iu)}\right\}\,du\right),\ x>0.$$

Let the function h be defined by

$$h(u)=\frac{d}{du}\left\{\frac{e^{-i\pi/6}\mathrm{Ai}\left(e^{-i\pi/6}u\right)}{i\mathrm{Ai}(iu)}\right\}.$$

Using Olver (2010)[2]:

$$\mathrm{Ai}\left(e^{-i\pi/6}u\right)=\mathrm{Ai}\left(e^{-2i\pi/3}iu\right)=\tfrac12e^{-i\pi/3}\left\{\mathrm{Ai}(iu)+i\mathrm{Bi}(iu)\right\},$$

we obtain

$$h(u)=\tfrac12\left\{\frac{\mathrm{Ai}'(iu)+i\mathrm{Bi}'(iu)}{i\mathrm{Ai}(iu)}-\frac{(\mathrm{Ai}(iu)+i\mathrm{Bi}(iu))\,\mathrm{Ai}'(iu)}{i\mathrm{Ai}(iu)^2}\right\},$$

and using the Wronskian $\mathrm{Ai}(z)\mathrm{Bi}'(z)-\mathrm{Ai}'(z)\mathrm{Bi}(z)=1/\pi$ we conclude that

$$h(u)=\frac{1}{2\pi\mathrm{Ai}(iu)^2}.$$

This gives the desired result. □

[1] http://dlmf.nist.gov/9

[2] http://dlmf.nist.gov/9.2.E11

Remark 2.1 Lemma 2.2 is in fact equivalent to relation (5.10) in Janson, Louchard and Martin-Löf (2010). The difference in the scaling constants is caused by the fact that they consider the maximum of $W(t) - \frac{1}{2}t^2$ instead of the maximum of $W(t) - t^2$ (see also section 3) and the fact that they integrate from $-\infty$ to ∞ (in that way also the imaginary part drops out). However, they arrive at this relation in a completely different way. So in this case we can go from Corollary 2.1 in Groeneboom (2010) to the result in Janson, Louchard and Martin-Löf (2010), just by using integration by parts. This might serve as a first step in establishing the relation between the representations in the two papers.

We are now ready to prove our main result. We will give two proofs, one based on the first equality in (2.3) and the other one based on the second equality.

Theorem 2.1 *Let N be defined by (1.1), and let f_N and F_N be the density and the distribution function of N, respectively. Then,*

(i)

$$f_N(x) \sim \frac{4\sqrt{x}}{3} \exp\left(-\frac{8x^{3/2}}{3\sqrt{3}}\right) \sum_{k=0}^{\infty} \frac{b_k}{x^{3k/2}}, \quad x \to \infty, \tag{2.4}$$

where the first coefficients are

$$b_0 = 1, \quad b_1 = \tfrac{19}{48}\sqrt{3}, \quad b_2 = -\tfrac{3851}{1536}, \quad b_3 = \tfrac{3380005}{221184}\sqrt{3}, \quad b_4 = -\tfrac{64744441455}{14155776}.$$

(ii)

$$1 - F_N(x) \sim \frac{1}{\sqrt{3}} \exp\left(-\frac{8x^{3/2}}{3\sqrt{3}}\right) \sum_{k=0}^{\infty} \frac{c_k}{x^{3k/2}}, \; x \to \infty,$$

where the first coefficients are

$$c_0 = 1, \quad c_1 = \tfrac{19}{48}\sqrt{3}, \quad c_2 = -\tfrac{4535}{1536}, \quad c_3 = \tfrac{3869785}{221184}\sqrt{3}, \quad c_4 = -\tfrac{7310315015}{14155776}.$$

Proof. Here we only derive the leading terms. Further terms in the asymptotic expansion of $f_N(x)$ are computed in the appendix, and those for $F_N(x)$ follow upon integrating the expansion for $f_N(x)$.

We start with the second representation in (2.3), and write

$$f_N(x) = \frac{1}{2^{1/3}\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Ai}\big(iu + 2^{2/3}x\big)}{\mathrm{Ai}(iu)^2}\, du = \frac{x}{2^{1/3}\pi i} \int_{-i\infty}^{i\infty} \frac{\mathrm{Ai}\big(x\left(u + 2^{2/3}\right)\big)}{\mathrm{Ai}(xu)^2}\, du, \quad x > 0. \tag{2.5}$$

We need the well-known asymptotic behavior of the Airy function (see Olver (2010)[3]):

$$\mathrm{Ai}(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi} z^{1/4}}, \quad \zeta = \tfrac{2}{3} z^{3/2}, \quad z \to \infty, \quad |\mathrm{ph}\, z| < \pi. \tag{2.6}$$

It follows that the behavior of the ratio of the Airy functions is given by

$$\frac{\mathrm{Ai}\big(x\left(u + 2^{2/3}\right)\big)}{\mathrm{Ai}(xu)^2} \sim \frac{2\sqrt{\pi} x^{1/4} u^{1/2}}{(u + 2^{2/3})^{1/4}} \exp\left(\tfrac{2}{3} x^{3/2} \phi(u)\right), \quad xu \to \infty, \quad |\mathrm{ph}\, xu| < \pi, \tag{2.7}$$

[3]http://dlmf.nist.gov/9.7.E5

where

$$\phi(u) = 2u^{3/2} - (u + 2^{2/3})^{3/2}.$$

This function has a minimum at $u = c = \frac{1}{3}2^{2/3}$, and we shift the path of integration in the second integral in (2.5) to the path $\mathcal{P}$ parallel to the imaginary axis, and running from $c - i\infty$ to $c + i\infty$. A similar path was used in the proof of (ii) of Corollary 3.4 in Groeneboom (1989).

A local expansion at $u = c$ gives

$$\phi(u) = \phi(c) + \tfrac{1}{2}\phi''(c)(u-c)^2 + \mathcal{O}\left((u-c)^3\right),$$

where

$$\phi(c) = -\frac{4}{\sqrt{3}}, \quad \phi''(c) = \frac{9\sqrt{3}}{2^{10/3}}. \tag{2.8}$$

We find a first approximation of $f_N(x)$ by neglecting the $\mathcal{O}-$term in the local expansion of $\phi(u)$ and by taking $u = c$ in the factor in front of the exponential factor in (2.7). This gives

$$f_N(x) \sim \frac{x}{2^{1/3}\pi i}\frac{2\sqrt{\pi}x^{1/4}c^{1/2}}{(c+2^{2/3})^{1/4}} \exp\left(\tfrac{2}{3}x^{3/2}\phi(c)\right) \int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{1}{3}x^{3/2}\phi''(c)(u-c)^2\right)\, du. \tag{2.9}$$

Evaluating the integral:

$$\int_{c-i\infty}^{c+i\infty} \exp\left(\tfrac{1}{3}x^{3/2}\phi''(c)(u-c)^2\right)\, du = \frac{i\sqrt{\pi}\,2^{5/3}}{3^{3/4}x^{3/4}},$$

we find the requested result

$$f_N(x) \sim \frac{4\sqrt{x}}{3} \exp\left(-\frac{8x^{3/2}}{3\sqrt{3}}\right), \quad x \to \infty.$$

Upon integrating we obtain the result for the distribution function $F_N(x)$. □

For two-sided Brownian motion we get similarly:

Corollary 2.1 *Let M be defined by*

$$M = \max_{t\in\mathbb{R}}\{W(t) - t^2\},$$

where W is standard two-sided Brownian motion, originating from zero. and let f_M and F_M be the density and the distribution function of M, respectively. Then:

(i)

$$f_M(x) = 2f_N(x)F_N(x) \sim \frac{8\sqrt{x}}{3} \exp\left\{-\frac{8x^{3/2}}{3\sqrt{3}}\right\}, \ x \to \infty.$$

(ii)

$$1 - F_M(x) \sim \frac{2}{\sqrt{3}} \exp\left\{-\frac{8x^{3/2}}{3\sqrt{3}}\right\}, \ x \to \infty.$$

Proof. This follows from Corollary 2.2 of Groeneboom (2010), which gives the representation:

$$f_M(x) = 2f_N(x)F_N(x),\ x \geq 0.$$

□

3 Concluding remarks

As pointed out to us by Svante Janson, the result implies certain facts for the moments of the distribution. For example, applying Theorem 4.5 in Janson and Chassaing (2004) together with Theorem 2.1 of the present paper gives:

$$(EM^r)^{1/r} \sim \frac{1}{2}(3/2)^{1/3}(r/e)^{2/3}, r \to \infty.$$

Theorem 2.1 can also easily be extended to a result for

$$M_c = \max_{t \geq 0}\{W(t) - ct^2\},$$

by using the scaling relation:

$$M_c = c^{-1/3}M,$$

see (1.7) of Janson, Louchard and Martin-Löf (2010). So, for example, Theorem 2.1 implies:

$$\mathbb{P}\{M_c \geq x\} = \mathbb{P}\left\{M \geq c^{1/3}x\right\} \sim \frac{1}{\sqrt{3}}\exp\left\{-\frac{8x^{3/2}\sqrt{c}}{3\sqrt{3}}\right\}, \ x \to \infty.$$

Also, by Lemma 2.2, the density of M_c is given by:

$$f_{M_c}(x) = \frac{(4c)^{1/3}}{\pi}\,\mathrm{Re}\left(\int_0^\infty \frac{\mathrm{Ai}\big(iu + (4c)^{1/3}x\big)}{\mathrm{Ai}(iu)^2}\,du\right), \ x > 0. \tag{3.10}$$

4 Appendix. Computing more coefficients of the asymptotic expansions

For obtaining more coefficients in the asymptotic expansion of $f_N(x)$ and $F_N(x)$ we need more details of the asymptotic behavior of the Airy function. We have the well-known expansion (see Olver (2010)[4]):

$$\mathrm{Ai}(z) \sim \frac{e^{-\zeta}}{2\sqrt{\pi}z^{1/4}}\sum_{k=0}^{\infty}(-1)^k\frac{u_k}{\zeta^k}, \quad z \to \infty, \quad |\mathrm{ph}\, z| < \pi, \tag{4.11}$$

where

$$\zeta = \tfrac{2}{3}z^{3/2},$$

and

$$u_k = \frac{(2k+1)(2k+3)(2k+5)\cdots(6k-1)}{(216)^k\,k!} = \frac{\Gamma(3k+\frac{1}{2})}{54^k\,k!\,\Gamma(k+\frac{1}{2})},$$

The first coefficients are

$$u_0 = 1, \quad u_1 = \tfrac{5}{72}, \quad u_2 = \tfrac{385}{10368}, \quad u_3 = \tfrac{85085}{2239488}, \quad u_4 = \tfrac{37182145}{644972544}.$$

Using (2.7) we write the second integral representation of $f_N(x)$ in (2.5) in the form

$$f_N(x) = \frac{2^{2/3}x^{5/4}}{\sqrt{\pi}\,i}\left(\frac{c}{4}\right)^{1/4}\int_{c-i\infty}^{c+i\infty}\exp\left(\tfrac{2}{3}x^{3/2}\phi(u)\right)A(u,x)\,du, \quad c = \tfrac{1}{3}2^{2/3}, \tag{4.12}$$

[4] http://dlmf.nist.gov/9.7.E5

where $A(u,x)$ is a slowly varying function along the path of integration. We have the asymptotic representation

$$A(u,x) \sim \left(\frac{c}{4}\right)^{-1/4} \frac{u^{1/2}}{(u+2^{2/3})^{1/4}} \frac{\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{u_k}{\zeta^k}}{\left(\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{u_k}{\eta^k}\right)^2}, \tag{4.13}$$

where

$$\zeta = \xi\left(u+2^{2/3}\right)^{3/2}, \quad \eta = \xi u^{3/2}, \quad \xi = \tfrac{2}{3}x^{3/2}. \tag{4.14}$$

The path of integration in (4.12) cuts the real $u-$axis at the saddle point of the exponential function. For large u we have $\phi(u) \sim u^{3/2}$, and for $u \to \pm i\infty$ we have $\operatorname{ph}\phi(u) \sim \pm 3\pi/4$. This shows the rate of convergence along this path. The optimal rate of convergence occurs when we take the saddle point contour, or path of steepest descent, defined by $\Im\phi(u) = \Im\phi(c) = 0$. At infinity along this path we have $\operatorname{ph} u = \pm 2\pi/3$. Along the saddle point contour the quantities η and ζ are large, because the variable of integration is bounded away from the origin. Also, η and ζ have suitable phases for using the asymptotic expansions as shown in (4.13).

By manipulating the asymptotic series, we obtain the following expansion:

$$A(u,x) \sim \left(\frac{c}{4}\right)^{-1/4} \frac{u^{1/2}}{(u+2^{2/3})^{1/4}} \sum_{k=0}^{\infty} \frac{A_k(u)}{\xi^k}, \tag{4.15}$$

where the first few coefficients are given by

$$\begin{aligned}
A_0(u) &= 1,\\
A_1(u) &= \frac{-5R+10}{72u^{3/2}},\\
A_2(u) &= \frac{385R^2-620-100R}{10368u^3},\\
A_3(u) &= \frac{9300R+138520-85085R^3+11550R^2}{2239488u^{9/2}},\\
A_4(u) &= \frac{-2770400R-62797040+37182145R^4-1432200R^2-3403400R^3}{644972544u^6},
\end{aligned} \tag{4.16}$$

where

$$R = \frac{u^{3/2}}{(u+3c)^{3/2}}.$$

Substituting the expansion in (4.15) into (4.12) we obtain

$$f_N(x) \sim \frac{2^{2/3}x^{5/4}}{\sqrt{\pi}\, i} \left(\frac{c}{4}\right)^{1/4} \sum_{k=0}^{\infty} \frac{\Phi_k(\xi)}{\xi^k}, \tag{4.17}$$

where

$$\Phi_k(\xi) = \left(\frac{c}{4}\right)^{-1/4} \int_{\infty e^{-2\pi i/3}}^{\infty e^{2\pi i/3}} \frac{u^{1/2}}{(u+2^{2/3})^{1/4}} \exp\left(\xi\phi(u)\right) A_k(u)\, du. \tag{4.18}$$

Next we transform these integrals into a standard form by putting

$$\phi(u) - \phi(c) = \tfrac{1}{2}\phi''(c)v^2, \tag{4.19}$$

Table 1: The first coefficients $C_j^{(k)}$ defined in (4.22).

k	$C_0^{(k)}$	$C_1^{(k)}$	$C_2^{(k)}$	$C_3^{(k)}$	$C_4^{(k)}$
0	1	$\frac{79}{1152}\sqrt{3}$	$-\frac{100031}{884736}$	$\frac{923668975}{3057647616}\sqrt{3}$	$-\frac{18376891706495}{4696546738176}$
1	$\frac{25}{128}\sqrt{3}$	$-\frac{29165}{49152}$	$\frac{180113185}{113246208}\sqrt{3}$	$-\frac{2860100405525}{130459631616}$	$\frac{85377203586646825}{601157982486528}\sqrt{3}$
2	$-\frac{13365}{32768}$	$\frac{8404125}{4194304}\sqrt{3}$	$-\frac{40293630095}{1073741824}$	$\frac{368687253142325}{1236950581248}\sqrt{3}$	$-\frac{49545968934273638575}{5699868278390784}$
3	$\frac{2649065}{4194304}\sqrt{3}$	$-\frac{36271635085}{1610612736}$	$\frac{945966020191985}{3710851743744}\sqrt{3}$	$-\frac{40394551713291361525}{4274901208793088}$	$\frac{2567961709695678056917625}{19698744770118549504}\sqrt{3}$
4	$-\frac{9582104685}{2147483648}$	$\frac{80522162743295}{824633720832}\sqrt{3}$	$-\frac{3320764894803103375}{633318697598976}$	$\frac{204107643936013882342175}{2188749418902061056}\sqrt{3}$	$-\frac{17534973455328077403049668175}{3361919107433565782016}$

where $\phi(c)$ and $\phi''(c)$ are given in (2.8). We prescribe $\text{sign}(u-c) = \text{sign}(v)$, and we need the coefficients in the expansion

$$u = c + v + d_2 v^2 + d_3 v^3 + \ldots,$$

of which the first few are given by

$$d_2 = \frac{5}{48c}, \quad d_3 = -\frac{1}{72c^2}, \quad d_4 = \frac{115}{27648c^3}, \quad d_5 = -\frac{385}{221184c^4}, \quad d_6 = \frac{1705}{1990656c^5}.$$

The transformation gives

$$\Phi_k(u) = \exp\left(\xi\phi(c)\right) \int_{-i\infty}^{i\infty} B_k(v) \exp\left(\tfrac{1}{2}\phi''(c)\xi v^2\right)\, dv, \tag{4.20}$$

where

$$B_k(v) = \left(\frac{c}{4}\right)^{-1/4} \frac{u^{1/2}}{(u+3c)^{1/4}} A_k(u)\, \frac{du}{dv}. \tag{4.21}$$

The final step is to expand each $B_k(v)$:

$$B_k(v) = \sum_{j=0}^{\infty} B_j^{(k)} v^j,$$

to obtain the expansions for $\Phi_k(\xi)$:

$$\Phi_k(\xi) \sim i\sqrt{\frac{2\pi}{\phi''(c)\,\xi}} \exp\left(\xi\phi(c)\right) \sum_{j=0}^{\infty} \frac{C_j^{(k)}}{\xi^j}, \quad C_j^{(k)} = \frac{B_{2j}^{(k)}(-1)^j 2^j \left(\frac{1}{2}\right)_j}{(\phi''(c))^j}, \tag{4.22}$$

and putting these expansions into (4.17). This gives

$$f_N(x) \sim \frac{4\sqrt{x}}{3} \exp\left(-\frac{8x^{3/2}}{3\sqrt{3}}\right) \sum_{k=0}^{\infty} \frac{D_k}{\xi^k}, \quad D_k = \sum_{j=0}^{k} C_j^{(k-j)},$$

and the b_k of (2.4) are given by

$$b_k = \left(\tfrac{3}{2}\right)^k D_k, \quad k = 0, 1, 2, \ldots.$$

Acknowledgements We thank the referees for their careful reading and useful comments. We also want to thank Svante Janson and Jon Wellner for their useful comments and Zakhar Kabluchko for bringing Hüsler and Piterbarg (1999) to our attention. Nico M. Temme acknowledges financial support from *Ministerio de Educación y Ciencia*, project MTM2006–09050, *Ministerio de Ciencia e Innovación*, project MTM2009-11686, and *Gobierno of Navarra*, Res. 07/05/2008.

References

Drazin, P. G. and Reid, William Hill, *Hydrodynamic Stability.* Cambridge University Press, Cambridge, UK, 1981.

Groeneboom, P. (1989) Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields*, **81**, 31-41.

Groeneboom, P. (2010). The maximum of Brownian motion minus a parabola. *Electronic Journal of Probability*, **15**, 1930-1937.

Groeneboom, P. and Jongbloed, G. (2010). Testing monotonicity of a hazard: asymptotic distribution theory. Submitted.

Hüsler, J. and Piterbarg, V. (1999). Extremes of a certain class of Gaussian processes. *Stochastic Processes and their Applications*, **83**, 257-271.

Janson, S. and Chassaing, P. (2004). The center of mass of the ISE and the Wiener index of trees. *Electronic Comm. Probab.*, **9**, 178-187.

Janson, S., Louchard, G. and Martin-Löf, A. (2010). The maximum of Brownian motion with a parabolic drift. *Electronic Journal of Probability*, **15**, 1893-1929.

Olde Daalhuis, A.B. (2010). *Hypergeometric Function.* Chapter 15 of *NIST Handbook of Mathematical Functions.* Cambridge University Press, Cambridge, UK, 2010.

Olver, F.W.J. (2010). *Airy and Related Functions.* Chapter 9 of *NIST Handbook of Mathematical Functions.* Cambridge University Press, Cambridge, UK, 2010.